Mathematical analysis of a seawater intrusion problem. Application to the hydraulic conductivity identification

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Modeling:

- Fundamental laws; Hypothesis on the fluid and on the medium
- Upscaling procedure

Mathematical analysis

- Existence in the case of sharp interface approach
- Existence in the case of sharp-diffuse interface approach
- Regularity result and Uniqueness

Numerical simulations

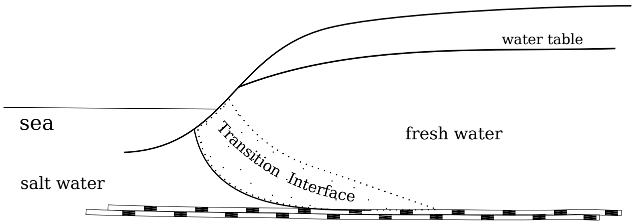
- P_1 Lagrange finite element method
- Comparison between FEM and FVM

Hydraulic conductivity identification

- Existence and characterization of an optimal control
- Numerical simulations

Free and confined aquifer

- Confined aquifer : The lower and upper surfaces of the aquifer are impermeables.
- Free aquifer : The upper surface is constituted with a permeable This aquifer are rechargeable in water with the raining falls but more sensitive to the pollution problem. Example : The phréatic sheet.



The classical Darcy law for porous media gives

$$q = -K\,\nabla(\rho g H), \qquad K = \frac{\kappa}{\mu}$$

where K is the hydraulic conductivity and

- $H = \frac{P}{\rho g} + z$ the hydraulic head,
- q The Darcy's flux,
- ρ : the volumic mass of the fluid,
- g : the standard gravity constant,
- μ : the dynamic viscosity of the fluid,
- κ: the permeability tensor of the porous medium.

- The conservation of mass is given by the following equation :

$$\frac{\partial(\phi\rho)}{\partial t} + \nabla \cdot (\rho q) = \rho Q,$$

where ϕ is the porosity of the medium and Q denotes a generic source term (for production and replenishment).

- First consequence of the low compressibility of the fluid combined with the low mobility of fluid leads to the following simplification in the mass conservation equation :

$$\rho\partial_t\phi + \phi\partial_t\rho + \rho\nabla \cdot q = \rho Q.$$

Hypothesis on the fluid and on the medium

Including $\frac{\partial_t \rho}{\rho} = \alpha_P \partial_t P$ and $\frac{\partial_t \phi}{(1-\phi)} = \beta_P \partial_t P$ in the mass conservation law, we get

$$\rho\big((1-\phi)\beta_P + \phi\alpha_P\big)\partial_t P + \rho\nabla\cdot q = \rho Q.$$

Using the hydraulic head and the Darcy law combined to $\rho > 0$, we finally get

$$S_0 \partial_t H - \nabla\cdot(K\nabla H) = Q \quad \text{where} \quad S_0 = \rho_0 g\big((1-\phi)\beta_P + \phi\alpha_P\big).$$

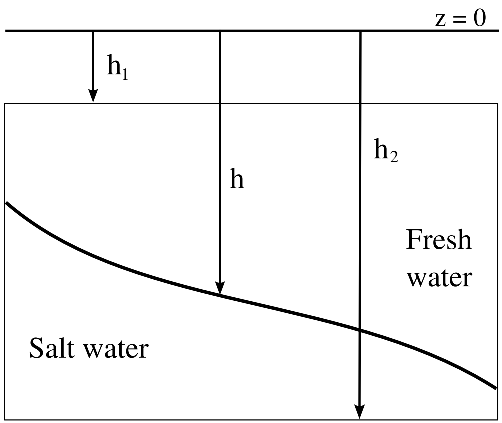
The fluid storage coefficient S_0 characterizes the workable water volume. It accounts for the rock and fluid compressibility.
We assume that fresh and salt water *and* the soil are weakly compressible. It means that the densities of the fluids and the porosity of the medium weakly depend on the pressure variations

$$\alpha_P \ll 1, \quad \beta_P \ll 1.$$

Hypothesis on the flow

- Sharp Interface Assumption : The thickness of the transition zone is assumed to be small in comparison of the aquifer's dimensions then the Freshwater and Saltwater are supposed to be immiscible fluids.
  Each liquid is confined to a well defined portion of the flow domain with an abrupt interface separating the two domains, called **sharp interface**.
- Dupuit Approximation Dupuit assumption consists in considering that the hydraulic head is constant along each vertical direction (vertical equipotentials). It is legitimate since one actually observes quasi-horizontal displacements when the thickness of the aquifer is small compared to its width and its length and when the flow is far from sinks and wells. This approximation is exact in the case of an homogeneous, isotropic and confined aquifer with constant thickness.

Notations



We now use these approximations to vertically integrate previous equations, thus reducing the 3D problem to a 2D problem.

- In the freshwater zone:
  Integrating the mass conservation law for the freshwater between the upper surface h_1 and the depth of the interface h:

$$\int_{h}^{h_1} \nabla \cdot q_f + S_{0f} \frac{\partial H_f}{\partial t} dz = 0$$

The Leibnitz's formulae yields :

$$\int_{h}^{h_1} S_{0f} \frac{\partial H_f}{\partial t} dz = S_{0f} \frac{\partial}{\partial t} \int_{h}^{h_1} H_f dz + S_{0f} H_f(h) \frac{\partial h}{\partial t} - S_{0f} H_f(h_1) \frac{\partial h_1}{\partial t}$$

The vertically averaged hydraulic head $\tilde{H}_f$ is defined by:

$$\tilde{H}_f = \frac{1}{B_f} \int_h^{h_1} H_f \, dz$$

with $B_f = h_1 - h$ the thickness of the freshwater zone, then

$$\int_h^{h_1} S_{0f} \frac{\partial \tilde{H}_f}{\partial t} dz = S_{0f} B_f \frac{\partial \tilde{H}_f}{\partial t}$$

Integrating the second term, we get :

$$\int_h^{h_1} \nabla \cdot q_f \, dz = \nabla \cdot (B_f \tilde{q}_f) + q_f(h_1) \cdot \nabla(z - h_1) - q_f(h) \cdot \nabla(z - h)$$

- Integrating the Darcy law, we obtain the specific flux :

$$\tilde{q}_f = \tilde{K}_f \nabla \tilde{H}_f \quad \text{and} \quad \tilde{q}_s = \tilde{K}_s \nabla \tilde{H}_s.$$

- In the freshwater zone:

$$S_{0f} B_f \frac{\partial \tilde{H}_f}{\partial t} = \nabla \cdot (B_f \tilde{K}_f \nabla \tilde{H}_f) - q_f(h_1) \cdot \nabla(z - h_1) + q_f(h) \cdot \nabla(z - h),$$

- In the saltwater zone, we integrate between the depth of the interface h and the lower surface h_2 (with $B_s = h - h_2$ the thickness of the saltwater zone) :

$$S_{0s} B_s \frac{\partial \tilde{H}_s}{\partial t} = \nabla \cdot (B_s \tilde{K}_s \nabla \tilde{H}_s) + q_s(h_2) \cdot \nabla(z - h_2) - q_s(h) \cdot \nabla(z - h).$$

Continuity equations

- Continuity of the pressure at the interface: $P_f = P_s$
  But $P_f = (\tilde{H}_f - h)\rho_f g$ and $P_s = (\tilde{H}_s - h)\rho_s g$

  Then $h = (1+\alpha)\tilde{H}_s - \alpha\tilde{H}_f$ with $\alpha = \frac{\rho_f}{\rho_s - \rho_f}$

- Continuity of the normal component of the velocity at the interface
  The free interface may be represented by a surface

$$F(x, y, z, t) = z - h(x, y, t) = 0.$$

By assuming a sharp interface, we suppose no diffusion across the interface between fresh and salt water that is

$$\left(\frac{\vec{q_f}}{\phi} - \vec{v}\right).\vec{n} = \left(\frac{\vec{q_s}}{\phi} - \vec{v}\right).\vec{n} = 0,$$

where $\vec{v}$ denotes the velocity of the interface, $\vec{n}$ the normal unit vector to the interface.

Continuity equations

Combining the continuity of the normal component of the velocity with the equation which governs the displacement of the interface $\partial_t F + \vec{v}.\nabla F = 0$, we get

- Flux at $z = h$:

$$\vec{q_f}(h) \cdot \nabla(z-h) = \vec{q_s}(h) \cdot \nabla(z-h) = \phi \frac{\partial h}{\partial t}$$

- Flux at $z = h_1$ (Case of Free aquifer)

$$F_1(x,y,z,t) = z - h_1(x,y,t) = 0.$$

$$\vec{q_f}(h_1) \cdot \nabla(z-h_1) = \phi \frac{\partial h_1}{\partial t}$$

- Flux at $z = h_2$

$$\vec{q_s}(h_2) \cdot \nabla(z-h_2) = 0$$

$2D$ Sharp Interface model

- $2D$ Sharp interface model

$$(S_{0f}B_f + \phi\beta)\frac{\partial \tilde{H}_f}{\partial t} - \phi\frac{\partial h}{\partial t} = \nabla\cdot(B_f K_f \nabla \tilde{H}_f) + Q_f,$$

$$S_{0s}B_s\frac{\partial \tilde{H}_s}{\partial t} + \phi\frac{\partial h}{\partial t} = \nabla\cdot(B_s K_s \nabla \tilde{H}_s) + Q_s.$$

with $\beta = 0$ for confined aquifer and $\beta = 1$ for free aquifer.

- By using that $(1+\alpha)\tilde{H}_s = \alpha\,\tilde{H}_f + h$, and $\tilde{H}_f = h_1$, we get :

$$\phi\frac{\partial h}{\partial t} - \nabla\cdot(\alpha K(h-h_2)\nabla h) - \nabla\cdot(K(h-h_2)\nabla h_1) = Q_s,$$

and

$$(S_{0f}B_f + \phi)\frac{\partial h_1}{\partial t} + \nabla(K(h_2-h_1)\nabla h_1) + \nabla\cdot(\alpha K(h-h_2)\nabla h) = Q_f + Q_s$$

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Global existence in degenerate case

J. Alkhayal, S. Issa, M. Jazar, R. Monneau (2016)

The seawater intrusion can be written as follows:

$$\begin{cases} \partial_t u_1 - \nabla \cdot \left(\nu\, u_1 \nabla \left(u_1 + u_2 \right)\right) & = 0 \quad \text{in} \quad \Omega_T, \\ \partial_t u_2 - \nabla \cdot \left(u_2 \nabla (\nu\, u_1 + u_2)\right) & = 0 \quad \text{in} \quad \Omega_T. \end{cases}$$

where

- $u_1 = h - h_1$ denotes the thickness of the freshwater part in the aquifer,
- $u_2 = h_2 - h$ denotes the thickness of the saltwater part in the aquifer,
- $\nu \in (0,1)$ is the relative density contrast between freshwater and saltwater.

Global existence in the degenerate case

J. Alkhayal, S. Issa, M. Jazar, R. Monneau (2016)

Definition of the nonnegative entropy function Ψ:

$$\Psi(a) - \frac{1}{e} = \begin{cases} a\,ln(a) & \text{if } a > 0, \\ 0 & \text{if } a = 0, \\ +\infty & \text{if } a < 0. \end{cases}$$

Theorem :

Let $u_{0,i} \geq 0, i = 1,2$ satisfying $\int_\Omega \Psi(u_{0,1}) + \int_\Omega \Psi(u_{0,2}) < +\infty$. Then there exists a weak solution $u = (u_1, u_2) \in \left(L^2(0,T;H^1(\Omega)) \cap C([0,T];(W^{1,\infty})')\right)^2$ with $u_i \geq 0$, in $\Omega_T.i = 1,2$. Moreover the solution satisfies the following entropy estimate :

$$\int_\Omega \Psi(u_1(t_2)) + \int_\Omega \Psi(u_2(t_2)) + \int_{t_1}^{t_2} \delta_0(|\nabla u_1|^2 + |\nabla u_2|^2) \leq \int_\Omega \Psi(u_{0,1}) + \int_\Omega \Psi(u_{0,2})$$

Proof

- Discretization in time:

$$\frac{u_{1,n+1} - u_{1,n}}{\Delta t} = div\big(\alpha\, u_{1,n+1}(\nabla u_{1,n+1} + \nabla u_{1,n+1})\big)$$
$$\frac{u_{2,n+1} - u_{2,n}}{\Delta t} = div\big(u_{2,n+1}(\alpha\,\nabla u_{1,n+1} + \nabla u_{1,n+1})\big)$$

- Truncation and Regularization : Let $\eta, \delta > 0$ and $0 < \epsilon < 1 < \ell$

$$\frac{u_{1,n+1} - u_{1,n}}{\Delta t} = div\big(\alpha\, T_{\epsilon,\ell}(u_{1,n+1})(\nabla\rho_\eta * \rho_\eta * u_{1,n+1} + \nabla\rho_\eta * \rho_\eta * u_{1,n+1} + \delta\, u_{1,n+1})\big)$$
$$\frac{u_{2,n+1} - u_{2,n}}{\Delta t} = div\big(T_{\epsilon,\ell}(u_{2,n+1})(\alpha\,\nabla\rho_\eta * \rho_\eta * u_{1,n+1} + \nabla\rho_\eta * \rho_\eta * u_{1,n+1} + \delta\, u_{1,n+1})\big)$$

where $T_{\epsilon,\ell}$ is truncation operator defined as $T_{\epsilon,\ell}(u) = u$ on (ϵ, ℓ) and is extended constantly and continuously outside (ϵ, ℓ).

Case of sharp-diffuse interface approach

Cahn, Hilliard (1958)
We introduce the two phases field

$$F = \begin{cases} 0 & \text{in fresh water .} \\ \frac{c_s}{2} & \text{on sharp interface .} \\ c_s & \text{in salt water.} \end{cases}$$

Function F satisfies an equation of Allen- Cahn type with three points of stability:

$$\partial_t F + \vec{v}.\nabla F - \delta \triangle F + \frac{F(F - \frac{c_s}{2})(F - c_s)}{\delta} = 0$$

where δ is the thickness of the diffuse interface. The previous equation becomes

$$-\phi \partial_t h + \vec{v}.\nabla(z - h) + \delta \triangle h = 0.$$

Sea Intrusion problem with sharp-diffuse interface approach

C. Choquet, M. Dhiédhiou, J. Li, C.R. (2015, 2016, 2017)

$$\phi\partial_t h - \nabla\cdot\big(KT_s(h)\nabla h\big) - \nabla\cdot\big(\delta\nabla h\big) - \nabla\cdot\big(KT_s(h)\nabla h_1\big) = -Q_s,$$

$$\phi\partial_t h_1 - \nabla\cdot\Big(K\big(T_f(h-h_1)+T_s(h)\big)\nabla h_1\Big) - \nabla\cdot\big(\delta\nabla h_1\big) - \nabla\cdot\big(KT_s(h)\nabla h\big) = -Q_f - Q_s.$$

General Cross-Diffusion system

$$\partial_t u_i = \sum_{k=1}^{N} \frac{\partial}{\partial x_k}\Big(\sum_{j=1}^{m} K_{i,j}^{k}(u) \frac{\partial u_j}{\partial x_k}\Big) =: \nabla \cdot J_i, \quad \text{in } \Omega_T, \qquad \text{for } i = 1, ..., m, \qquad (1)$$

- **Population dynamics** (N=m=2) (R. Redlinger (1995), L. Chen, A. Jungel (2004, 2006); T. Lepoutre, M. Pierre, G. Rolland (2012); S. Kouachi, K.E. Yong, R.D. Parshadz (2014) ...)

$$\begin{aligned} K_{1,1}^1(u) &= K_{1,1}^2(u) = (\alpha_{11}+\alpha_{12})\,u_2 + \delta, \\ K_{1,2}^1(u) &= K_{1,2}^2(u) = (\alpha_{11}+\alpha_{12})\,u_1, \\ K_{2,1}^1(u) &= K_{2,1}^2(u) = (\alpha_{21}+\alpha_{22})\,u_2, \\ K_{2,2}^1(u) &= K_{2,2}^2(u) = (\alpha_{21}+\alpha_{22})\,u_1 + \delta. \end{aligned}$$

  The unknown u_i, for $i = 1, 2$, stands for the population density of the i-th species.
- **Electrochemistry context** (Y. S. Choi, Z. Huan, R. Lui, 2003) The components of the matrices $K^k(u) = (K_{i,j}^k(u))_{1\le i,j\le m}$ are continuous in u and uniformly bounded with respect to u.

Global in time existence

C. Choquet, C. R., L. Rosier

$$\partial_t u_i - \nabla \cdot \Big(\delta_i \nabla u_i + T_l(u_i) \sum_{j=1}^{2} K_{i,j} \nabla u_j \Big) = 0 \quad \text{in } \Omega_T, \ \text{for } i = 1, 2. \tag{2}$$

with $T_l(u) = u$ continuously and constantly extended outside the interval $[0, l]$.

Theorem 3:

Assume that the tensor satisfy:

$$\frac{K_{1,2}^2}{K_{1,1}} < \frac{4\delta_2}{l}, \ \frac{K_{2,1}^2}{K_{2,2}} < \frac{4\delta_1}{l}.$$

Pick $u_i^0 \in L^2(\Omega)$ with $0 \leq u_i^0$ a.e. in Ω. Then for any $T > 0$, the problem (2) admits a weak solution $(u_i)_{i=1,2} \in W(0,T)^2$. Furthermore, the following maximum principle holds true:

$$0 \leq u_i(t,x) \quad \text{for a.e. } x \in \Omega, \ \text{for all } t \in (0,T) \text{ and for all } i = 1, 2.$$

Proof :

Step 1. Existence of a solution of the linearized system :

Definition of the map $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$

For the fixed point strategy, we define an application $\mathcal{F} : W(0,T)^2 \to W(0,T)^2$ by

$$\mathcal{F}(\bar{u}_1, \bar{u}_2) = \big(\mathcal{F}_1(\bar{u}_1, \bar{u}_2), \mathcal{F}_2(\bar{u}_1, \bar{u}_2)\big) = (u_1, u_2),$$

where (u_1, u_2) is the solution of the following initial boundary value problem

$$\partial_t u_1 - \nabla \cdot \Big(\delta_1 \nabla u_1 + T_l(\bar{u}_1) K_{1,1} \nabla u_1 + T_l(\bar{u}_1) K_{1,2} \nabla \bar{u}_2\Big) = 0, \quad (3)$$

$$\partial_t u_2 - \nabla \cdot \Big(\delta_2 \nabla u_2 + T_l(\bar{u}_2) K_{2,2} \nabla u_2 + T_l(\bar{u}_2) K_{2,1} \nabla \bar{u}_1\Big) = 0, \quad (4)$$

$$(u_1, u_2) = (0,0) \quad \text{in } (0,T) \times \Gamma, \quad (5)$$

$$(u_1(0,x), u_2(0,x)) = (u_1^0(x), u_2^0(x)) \quad \text{in } \Omega. \quad (6)$$

Proof :

- **Step 1**
  $\mathcal{F}$ is sequentially continuous in $(L^2(0,T;H))^2$.
- **Step 2**
  Existence of a convex and bounded subset of $(L^2(0,T;H))^2$, $\mathcal{C}$ such that $\mathcal{F}(\mathcal{C}) \subset \mathcal{C}$

$$\mathcal{C} := \{(u_1,u_2) \in W(0,T)^2;\ (u_1(0,.),u_2(0,.)) = (u_1^0,u_2^0),$$
$$\delta_1\|\nabla u_1\|^2_{L^2(\Omega_T)} + \delta_2\|\nabla u_2\|^2_{L^2(\Omega_T)} \leq C_0, \|\partial_t u_1\|_{L^2(0,T,V')} \leq D_M,\ \|\partial_t u_2\|_{L^2(0,T,V')} \leq D'_M\}$$

  where
$$C_0 := \frac{1}{2}\int_\Omega u_1(0,x)^2\,dx + \frac{1}{2}\int_\Omega u_2(0,x)^2\,dx < +\infty,$$
  and $M := \max(\sqrt{C_0/\delta_1}, \sqrt{C_0/\delta_2})$. The sequential compactness of $\mathcal{F}_i(\mathcal{C})$ in $L^2(0,T;H)$ $(i=1,2)$ is straightforward due to the Aubin-Lions' lemma.
- **Step 3**
  Non negativity of the solutions: $0 \leq u_1(t,x)$ for a.e. $x \in \Omega$ and all $t \in (0,T)$.
  $(\to\ h_1(t,x) \leq h(t,x) \leq h_2)$

N. G. Meyers (1963), A. Bensoussan, J. L. Lions, G. Papanicolaou (1978).

Let $Au = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(A_{ij}(t,x)\frac{\partial u}{\partial x_j}\right)$. We assume that that there exists $\gamma > 0$ s.t. $\sum_{i,j=1}^{N} A_{i,j}(t,x)\xi_i\xi_j \geq \gamma|\xi|^2, \quad \forall x \in \Omega \text{ and } \xi \in \mathbb{R}^N$. We set $\beta := \max_{1\leq i,j\leq n} ||A_{i,j}||_{L^\infty(\Omega_T)}$.

Lemma :

Let $f \in L^2(0,T,H^{-1}(\Omega))$, $u^0 \in H$, and $u \in L^2(0,T;H_0^1(\Omega))$ be the solution of

$$\begin{cases} \frac{\partial u}{\partial t} + Au = f \quad in \quad \Omega_T, \\ u(0) = u^0. \end{cases}$$

Then there exists $p > 2$, depending on γ, β and Ω, such that if $u^0 \in W_0^{1,p}(\Omega)$ and $f \in L^p(0,T;W^{-1,p}(\Omega))$, then $u \in L^p(0,T;W_0^{1,p}(\Omega))$. Furthermore, there exists a constant $C(\gamma,\beta,p) > 0$ such that

$$||u||_{L^p(0,T,W_0^{1,p}(\Omega))} \leq C(\gamma,\beta,p)(||f||_{L^p(0,T;W^{-1,p}(\Omega))} + ||u^0||_{W_0^{1,p}(\Omega)}). \tag{7}$$

Regularity result

- We denote

$$g(p) = ||P^{-1}||_{\mathcal{L}(Y_p;X_p)},$$

where $P = \dfrac{\partial}{\partial t} - \Delta$, associated to homogeneous Dirichlet B. C. and

$$X_p = L^p(0,T;W_0^{1,p}(\Omega)) \text{ and } Y_p = L^p(0,T;W^{-1,p}(\Omega)),$$

- We apply the previous Lemma to tensors $A_i = (\delta_i + K_{i,i}T_l(\bar{u}_i))\mathcal{I}d$, $i = 1,2$, then $\gamma_i = \delta_i$, $\beta_i = \delta_i + l\,K_{i,i}$ for $i = 1,2$.
Hence, if $p > 2$ is such that

$$k_i(p) := g(p)(1 - \frac{\delta_i}{\delta_i + l\,K_{i,i}}) < 1, \ \ i = 1,2,$$

then this exponent p is convenient.
- $\mathcal{F}(\mathcal{D}) \subset \mathcal{D}$ where

$$\mathcal{D} := \{(u_1,u_2) \in [W(0,T) \cap L^p(0,T;W^{1,p}(\Omega))]^2, \ (u_1(0),u_2(0)) = (u_1^0,u_2^0); \\ \|(u_1;u_2)\|_{W(0,T)^2} \le A, \ \|\nabla u_1\|_{L^p(\Omega_T)} \le M' \text{ and } \|\nabla u_2\|_{L^p(\Omega_T)} \le M'\}.$$

Uniqueness

- The L^4-regularity of the gradient of the solution combined with Gagliardo-Nirenberg inequality for $p = 4$ allows to prove the uniqueness.

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Theorem 4:

We assume that the parameters (l, δ_1, δ_2) and the tensor K satisfy

$$K_{i,i} < \frac{1}{g(4)-1}\frac{\delta_i}{l},\ i = 1, 2,\ \frac{K_{1,2}^2}{K_{1,1}} < \frac{3\,\delta_2}{l} \quad \text{and} \quad \frac{K_{2,1}^2}{K_{2,2}} < \frac{3\,\delta_1}{l}.$$

If $(u_{1,0}, u_{2,0}) \in W^{1,4}(\Omega)^2$, then the solution (u_1, u_2) is unique in $W(0,T)^2$.

Finite element scheme.

If h_b^n et $h_{1,b}^n$ are in $(\mathcal{I}_b(h_D) + V_b^k) \times (\mathcal{I}_b(h_{1,D}) + V_b^k)$,

$$0 \leqslant h_{1,b}^n \leqslant h_b^n \leqslant h_2,$$

Semi-implicit in time scheme :
Find $(h_{1,b}^{n+1}, h_b^{n+1}) \in (\mathcal{I}_b(h_{1,D}) + V_b^k) \times (\mathcal{I}_b(h_D) + V_b^k)$, $\forall w \in V_b^k$.

$$\phi \frac{h_{1,b}^{n+1} - h_{1,b}^n}{\delta t} - \nabla.(\delta \nabla h_{1,b}^{n+1}) - \nabla.(T_f(h_b^n - h_{1,b}^n)\nabla h_{1,b}^{n+1})$$

$$-\nabla.(T_s(h_b^n))\nabla(h_{1,b}^{n+1} + h_b^n) = {Q_s}^{n+1} + {Q_f}^{n+1}$$

$$\phi \frac{h_b^{n+1} - h_b^n}{\delta t} - \nabla.(\delta \nabla h_b^{n+1}) - \nabla.(T_s(h_b^n)\nabla(h_b^{n+1} + h_{1,b}^{n+1})) = {Q_s}^{n+1}$$

Error estimates for FEM .

Theorem 4:

If $\left(\phi - \frac{2\,h_2^2 K_+}{\delta}(2\,K_+ + K_-)C(b)^2\delta t\right) > 0$, there exists a constant $C > 0$, s.t. for any solution (h, h_1) in $Y(\Omega_T) = C^2([0,T], L^2(\Omega)) \cap C^1([0,T], H^{k+1}(\Omega))$, $k \geq 1$. Moreover, we have

$$\max_{0\le n\le N} \|h(t^n) - h_b^n\|_{L^2} \le C\,(b^k + \delta t) max(\|h\|_{Y(\Omega_T)}, \|h_1\|_{Y(\Omega_T)}),$$

$$\max_{0\le n\le N} \|h_1(t^n) - h_{1,b}^n\|_{L^2} \le C\,(b^k + \delta t) max(\|h\|_{Y(\Omega_T)}, \|h_1\|_{Y(\Omega_T)}),$$

$$\left[\frac{1}{\delta t}\sum_{n=1}^{N} \|h(t^n) - h_b^n\|_{H^1}^2\right]^{\frac{1}{2}} \le C\,(b^k + \delta t) max(\|h\|_{Y(\Omega_T)}, \|h_1\|_{Y(\Omega_T)}),$$

$$\left[\frac{1}{\delta t}\sum_{n=1}^{N} \|h_1(t^n) - h_{1,b}^n\|_{H^1}^2\right]^{\frac{1}{2}} \le C\,(b^k + \delta t) max(\|h\|_{Y(\Omega_T)}, \|h_1\|_{Y(\Omega_T)}).$$

A finite volume scheme for this sea intrusion problem

Let $\mathcal{T}$ the rectangular mesh of the interval $(0, L_x) \times (0, L_y)$ consisting of $N \times M$ cells denoted by K_{ij} $1 \leq i \leq N$. $1 \leq j \leq M$, and $N \times M$ points of $(0, L_x) \times (0, L_y)$. $(x_i, y_j)_{1 \leq i \leq N, 1 \leq j \leq M}$ satisfying the following assumptions:
$K_{ij} = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times (y_{i-\frac{1}{2}}, y_{i+\frac{1}{2}})$ such that

$$x_0 = x_{\frac{1}{2}} = 0 < x_1 < x_{\frac{1}{2}} ..... < x_{i-\frac{1}{2}} < x_i < x_{i+\frac{1}{2}} < x_N < x_{N+\frac{1}{2}} = x_{N+1} = L_x,$$

$$y_0 = y_{\frac{1}{2}} = 0 < y_1 < y_{\frac{1}{2}} ..... < y_{i-\frac{1}{2}} < y_i < y_{i+\frac{1}{2}} < y_N < y_{N+\frac{1}{2}} = y_{N+1} = L_y.$$

We set $V(u) = -\frac{1}{2}(h_2 - u)^2$, $\lambda = T_s(u) = (h_2 - u)$.
The discrete unknowns $u^n_{i,j}$ are expected to be approximation of the value of $u^n(t_n, x_i, y_j)$ or of the mean value of u over K_{ij}).

A finite volume scheme for this sea intrusion problem

We consider a regular structured mesh i.e.

$$\begin{cases} h_i = h \ \forall \, i \in \{1, ..., N\} \\ k_j = k \ \forall \, j \in \{1, ..., M\} \end{cases}$$

and the point (x_i, y_j) is assumed to be the center of K_{ij} then $h_i^+ = h_i^- = \frac{h}{2}$ and $h_{i+\frac{1}{2}} = h$ (and the same for k), The explicit in time numerical scheme for the approximation of the equations is therefore

$$\frac{h_{ij}^{n+1} - h_{ij}^n}{\delta t} + \frac{1}{h^2}(V(h)_{i+1,j}^n - 2\,V(h)_{i,j}^n + \ V(h)_{i-1,j}^n)$$
$$+\frac{1}{k^2}(\ V(h)_{i,j+1}^n - 2\,V(h)_{i,j}^n + \ V(h)_{i,j-1}^n) =$$
$$-\frac{2}{h^2}\{\frac{\lambda_{i,j}^n \lambda_{i+1,j}^n}{\lambda_{i+1,j}^n + \lambda_{i,j}^n}(f_{i+1,j}^n - f_{i,j}^n) + \frac{\lambda_{i,j}^n \lambda_{i-1,j}^n}{\lambda_{i,j}^n + \lambda_{i-1,j}^n}(f_{i,j}^n - f_{i-1,j}^n)\}$$
$$-\frac{2}{k^2}\{\frac{\lambda_{i,j}^n \lambda_{i,j+1}^n}{\lambda_{i,j}^n + \lambda_{i,j+1}^n}(f_{i,j+1}^n - f_{i,j}^n) + \frac{\lambda_{i,j}^n \lambda_{i,j-1}^n}{\lambda_{i,j}^n + \lambda_{i,j-1}^n}(f_{i,j}^n - f_{i,j-1}^n)\} - {I_s}_{i,j}^n.$$

In the same way, the numerical scheme for the approximation of the equation for f is :

$$\frac{k}{h}(f^n_{i+1,j} - 2f^n_{i,j} + f^n_{i-1,j}) + \frac{h}{k}(f^n_{i,j+1} - 2f^n_{i,j} + f^n_{i,j-1})$$

$$= \frac{1}{h_b - h_t}(\frac{k}{h}(V(h)^n_{i+1,j} - 2\, V(h)^n_{i,j} + V(h)^n_{i-1,j})$$

$$+\frac{h}{k}(V(h)^n_{i,j+1} - 2\, V(h)^n_{i,j} + \dot{V}(h)^n_{i,j-1})) + hk(I_f + I_s)^n_{i,j} := B^n_{i,j}.$$

- Oulhaj, Ahmed Ait Hammou, Numerical analysis of a finite volume scheme for a seawater intrusion model with cross-diffusion in an unconfined aquifer. Numer. Methods Partial Differential Equations 34 (2018), no. 3, 857-880.

Analytical solution proposed by Keulegan (Confined case)

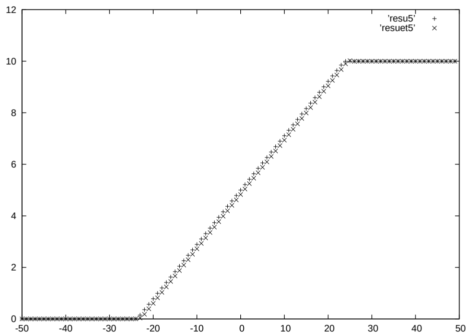
We consider a confined aquifer of uniform thickness with a vertical interface at $x = 0$, with salt water in the part $x < 0$ and freshwater in $x > 0$. At $t = 0$, the gate is removed and the interface begins to move owing to the density difference. The interface is described by a linear profile pivoting around a fixed point $(0, -D/2)$. Keulegan gave an analytic solution for the motion of the elevation of the interface

$$h(x,t) = -\frac{D}{2}\left(1 + \frac{x}{L(t)}\right).$$

The location of the toe (the intersection of the interface with the bottom of the aquifer) is given for $h = -D$ by

$$L(t) = \sqrt{\frac{D\alpha K t}{\phi}}$$

Interface evolution at time t=5,10 days with FEM, N=100.



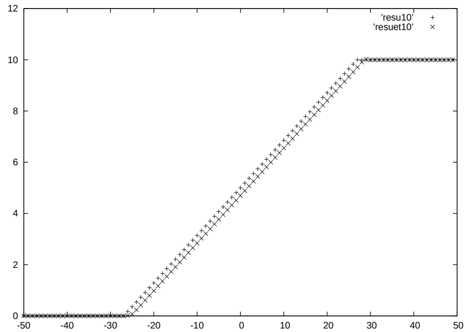
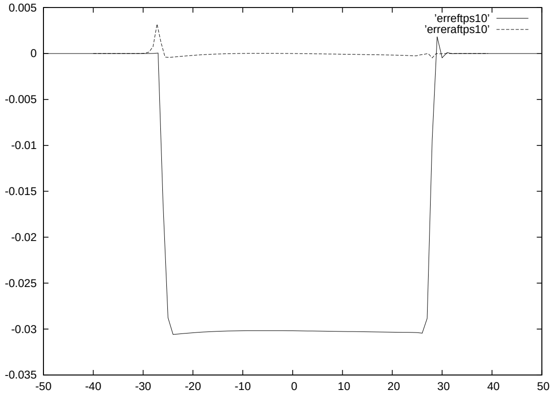
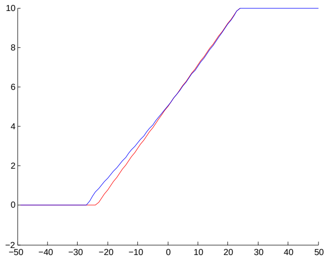


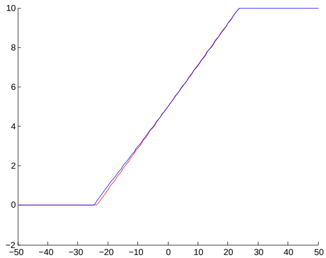
Figure : Comparison between FV solution and analytic solution derived from Keulegan model at time t=5 days (on the left) and t=10 days (on the right). N=100

Relative error for FEM with $N = 100$ and $N = 160$.



Interface evolution at time $t = 5$ days with FVM with $N = 100$ and $N = 160$.





Non confined case, with Dirichlet boundary conditions

We couple the problem with the tides effects. For this simulation we use the parameters in Cooper'64 after a rescaling to our small aquifer.

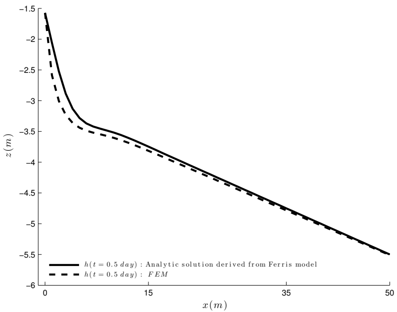
- H.H. Cooper, *A hypothesis concerning the dynamic balance of fresh water and salt water in a coastal aquifer*, U.S. Geological Survey Water-Supply Paper 1613-C, 1–12, 1964.

We impose a Dirichlet boundary condition on the left boundary $\{x = 0\}$ for the saltwater elevation h. Its value is computed with the classical tide-produced change model for the artesian head of Ferris'51. We compare the interface h obtained with FE method with a reference solution, here derived from the analytic formula of Ferris'51.

- J. G. Ferris, *Cyclic fluctuations of water level as a basis for determining aquifer transmissibility*, Int. Assoc. Sci. Hydrology Publ., Vol. 1, 97–101, 1951.

Non confined case

With Dirichlet boundary conditions



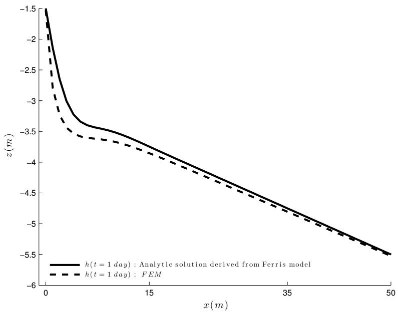
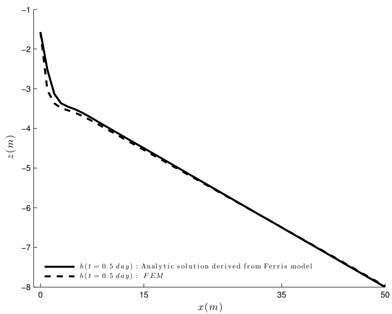


Figure : Comparison between FE solution and analytic solution derived from Ferris model. Times t=0,5 day (on the left) and t=1 day (on the right), N=100.

Non confined case



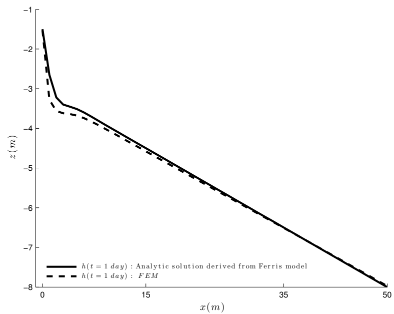


Figure : Comparison between FE solution and analytic solution derived from Ferris model. Times t=0,5 day (on the left) and t=1 day (on the right), N=200.

Non confined case

	L^2-norm		L^∞-norm	
	N = 100	N = 200	N = 100	N = 200
$\|E_h(t=0.5)\|$	0.4325	0.0120	0.2558	0.0220
$\|E_h(t=1)\|$	0.6811	0.1901	0.3869	0.0474

Table : Norms of the error E_h between the FE solution and the analytic solution of Ferris model.

Stability condition assumed in the convergence analysis

We emphasize that the physical and numerical parameters satisfy the stability assumption:

$$\left(\phi - \frac{2\,h_2^2 K_+}{\delta}(2\,K_+ + K_-)C(b)^2\delta t\right) > 0.$$

Indeed, in our case $K_+ = K_- = 4.516\,10^{-4}\;m/s$, $\phi = 0.3$ and $\delta = 1.5\;m$.
Besides $C(b) = \mathcal{O}\left(\frac{100}{50}\right)$ and $\delta t = 864\;s$, we get

$$\frac{2\,h_2^2 K_+}{\delta}(2\,K_+ + K_-)C(b)^2\delta t = \frac{4 \times 6 \times (4.516)^2 \times 864 \times 10^{-6}}{1.5} = 0.28193 < \phi = 0.3.$$

Non confined case, with Neumann boundary conditions

Neumann boundary conditions are now considered. First, we take $N = 100$ points on each direction of the aquifer and we take $\delta t = 0.01$ day.

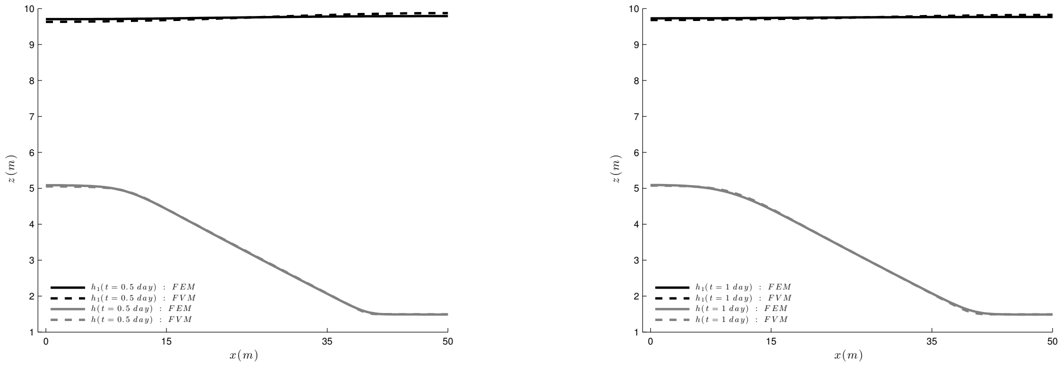


Figure : Comparison between FEM and FVM without forcing term. Times t=0,5 day (on the left) and t=1 day (on the right). This figure illustrates that FEM and FVM give similar qualitative results when the system evolves without forcing term.

Non confined case, with Neumann boundary conditions

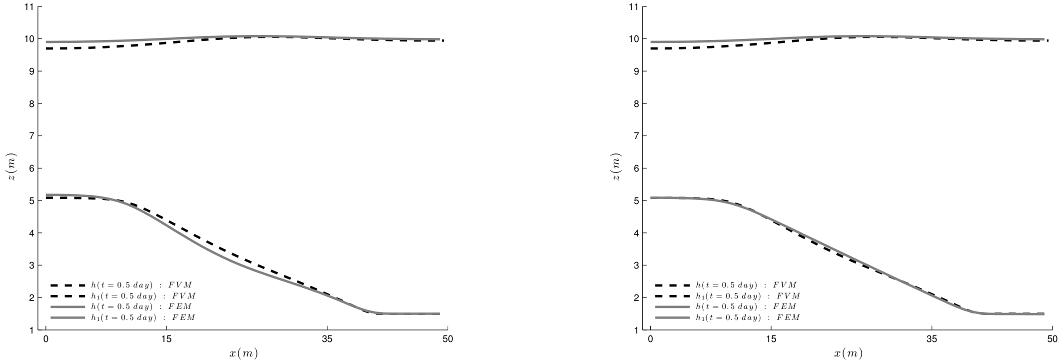


Figure : Comparison between FEM and FVM during a pumping process. N=100 (on the left) and N=200 (on the right).

Non confined case, with Neumann boundary conditions

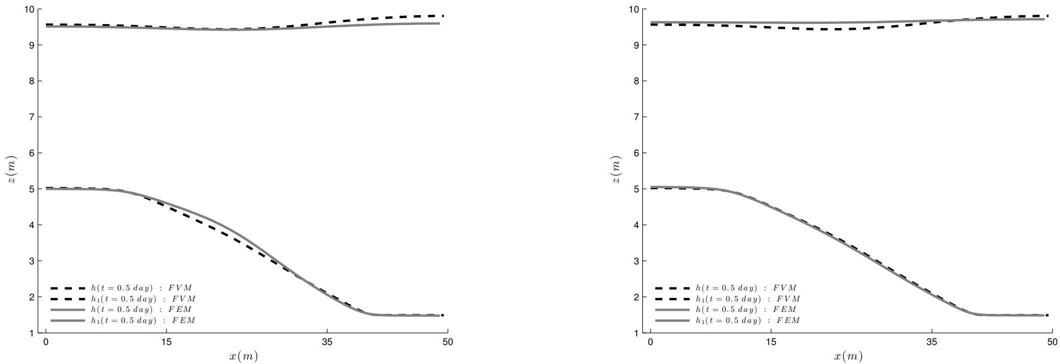


Figure : Comparison between FEM and FVM during an injection process. N=100 (on the left) and N=200 (on the right).

The control problem

The inverse problem is formulated by an optimization problem whose cost function measures the squared difference between experimental interfaces depths and those given by the model. We introduce the following control problem:

$$(\mathcal{O}) \begin{cases} \text{Find } K^* \in U_{adm} \text{ such that} \\ \mathcal{J}(K^*) = \inf_{K \in U_{adm}} \mathcal{J}(K), \end{cases}$$

with $\mathcal{J}(K) = \frac{1}{2}||h_1(K) - h_{1,obs}||^2_{L^2(\Omega_T)} + \frac{1}{2}||h(K) - h_{obs}||^2_{L^2(\Omega_T)}$, where $(h_1(K), h(K))$ is the weak solution of system $P(K)$ and $(h_{1,obs}, h_{obs})$ are the observed depths.
The set of admissible parameters U_{adm} is defined as a subset of $BV(\Omega)$:

$$U_{adm} = \{K \in BV(\Omega) \cap L^\infty(\Omega), K_m \leq K \leq K_M \text{ and } TV(K) \leq c\},$$

where K_m, K_M and c are nonnegative real constants.

Existence of optimal control

- M. H. Tber, M. E. Talibi, D. Ouaraza, Identification of the Hydraulic Conductivities in a Saltwater Intrusion Problem, J. Inv. Ill-Posed Problems 15, (2007).
- M. H. Tber, M. E. Talibi, D. Ouaraza, Parameters identification in a seawater intrusion model using adjoint sensitive method, Math. Comput. Simul. .77, pp 301-312 (2008).

Theorem

There exists at least one optimal control for the problem $(\mathcal{O})$.

Proof:

- U_{adm} is a compact subset of $L^2(\Omega)$
- Uniqueness of the solution of $P(K)$

Optimization problem

- We have now to determine the minimum of the cost function $\mathcal{J}$, the state system being the unsteady problem $P(K)$.
- We consider the system $P(K)$ as a constraint for the optimization problem.
- We use a Lagrangian functional $\mathcal{L}$ to derive the expression of an adjoint state allowing to get a workable expression of the cost functional derivative.

$$\begin{aligned}
\mathcal{L} \quad & (K, h_1, h, \lambda_f, \lambda_i) = \mathcal{J}(K) + \int_{\Omega_T} \phi \frac{\partial h}{\partial t} \lambda_i \, dxdt + \int_{\Omega_T} \phi \frac{\partial h_1}{\partial t} \lambda_f \, dxdt \\
+ \quad & \int_{\Omega_T} (\delta + \alpha K(x) T_s(h)) \nabla h \cdot \nabla \lambda_i \, dxdt + \int_{\Omega_T} \alpha \, K(x) \, T_s(h) \, \nabla h \cdot \nabla \lambda_f \, dxdt \\
+ \quad & \int_{\Omega_T} (\delta + K(x) \, T_s(h_1)) \, \nabla h_1 \cdot \nabla \lambda_f \, dxdt - \int_{\Omega_T} Q_s \lambda_i \, dxdt \\
+ \quad & \int_{\Omega_T} K(x) \, T_s(h) \, \nabla h_1 \cdot \nabla \lambda_i \, dxdt - \int_{\Omega_T} (Q_s + Q_f) \lambda_f \, dxdt. \qquad (8)
\end{aligned}$$

Optimization problem

The solution then corresponds to a saddle point of $\mathcal{L}$ considered as function of independent variables $h, h_1, \lambda_i, \lambda_f$ and K with λ_i and λ_f the Lagrange multipliers.
The minimum K^*, satisfy the following optimality system :

$$\left\{\begin{array}{ll} \frac{\partial \mathcal{L}}{\partial \lambda_i}(K^*, h_1^*, h^*, \lambda_f^*, \lambda_i^*) = 0, & \frac{\partial \mathcal{L}}{\partial \lambda_f}(K^*, h_1^*, h^*, \lambda_f^*, \lambda_i^*) = 0, \\ \frac{\partial \mathcal{L}}{\partial h}(K^*, h_1^*, h^*, \lambda_f^*, \lambda_i^*) = 0, & \frac{\partial \mathcal{L}}{\partial h_1}(K^*, h_1^*, h^*, \lambda_f^*, \lambda_i^*) = 0, \\ \frac{\partial \mathcal{L}}{\partial K}(K^*, h_1^*, h^*, \lambda_f^*, \lambda_i^*) \cdot (K - K^*) \geq 0, & \forall K \in U_{adm}, \end{array}\right.$$

with (h_1^*, h^*) the unique solution of $P(K^*)$ and $(\lambda_f^*, \lambda_i^*)$ the unique solution for $K = K^*$ of the following retrograde system $R(K)$

$$\left\{\begin{array}{l} -\phi \frac{\partial \lambda_f}{\partial t} - \nabla \cdot \left(\left(\delta + K(x)\, T_s(h_1) \right) \nabla \lambda_f \right) - \nabla \cdot \left(K(x) T_s(h) \nabla \lambda_i \right) \\ -K(x)\, \nabla h_1 \cdot \nabla \lambda_f = h_{1,obs} - h_1, \\ -\phi \frac{\partial \lambda_i}{\partial t} - \nabla \cdot \left(\delta \nabla \lambda_i + \alpha K T_s(h) \nabla (\lambda_f + \lambda_i) \right) - \alpha\, K(x) \nabla h \cdot \left(\nabla \lambda_i + \nabla \lambda_f \right) \\ -K(x)\, \nabla h_1 \cdot \nabla \lambda_i = h_{obs} - h, \\ \lambda_i = 0,\ \lambda_f = 0 \ \text{ on } \Gamma,\ \ \lambda_i(T, x) = \lambda_f(T, x) = 0,\ \ \forall x \in \Omega. \end{array}\right.$$

Existence and uniqueness for the adjoint problem

We performed the change of variable $t' = T - t$ in the retrograde system $R(K)$ in order to get initial conditions $\lambda_i(0,x) = 0$ and $\lambda_f(0,x) = 0$, $\forall x \in \Omega$.

Proposition:

Assume that

$$K_+ \leq \frac{\delta}{\alpha\, h_2}.$$

Let $(h_1(K), h(K))$ the solution of $P(K)$ with $K \in U_{adm}$, then the adjoint problem admits a unique weak solution.

Differentiability of the operator $\mathcal{Q}$

We introduce, $\mathcal{Q}$, the operator associating $(h(K), h_1(K))$, the solution of $P(K)$ with the hydraulic conductivity K and we check that $\mathcal{Q}$ is continuous and differentiable on suitable function spaces. For this purpose, we introduce an application, $\mathcal{R}$ allowing to implicitly define $\mathcal{Q}$ such that Then, $\forall(\varphi_i, \varphi_f) \in (L^2(0,T;H_0^1(\Omega)))^2$, we define $\mathcal{R}$ as follows

$$
\begin{aligned}
< \mathcal{R}(\bar{h}_1, \bar{h}, K), (\varphi_i, \varphi_f) >= & \int_{t_0}^{T}\int_\Omega \phi \frac{\partial h}{\partial t}\varphi_i \, dxdt + \int_0^T\int_\Omega \phi\frac{\partial h_1}{\partial t}\varphi_f \, dxdt \\
+ & \int_0^T\int_\Omega \left((\delta + \alpha K(x)\,(h_2 - h))\,\nabla h \cdot \nabla\varphi_i + K(x)\,(h_2 - h)\,\nabla h_1 \cdot \nabla \varphi_i\right) dxdt \\
+ & \int_0^T\int_\Omega \left((\delta + K(x)\,(h_2 - h_1))\,\nabla h_1 \cdot \nabla\varphi_f + \alpha\, K(x)\,(h_2 - h)\,\nabla h \cdot \nabla \varphi_f\right) dxdt \\
+ & \int_0^T\int_\Omega Q_s\varphi_i \, dxdt + \int_0^T\int_\Omega (Q_s + Q_f)\varphi_f \, dxdt.
\end{aligned}
$$

Differentiability of the operator $\mathcal{Q}$

The main point consists in finding the well-adapted function spaces so that the implicit function theorem is applicable. We define $\mathcal{R}$ s.t.

$$\begin{array}{rcl} Z(0,T)^2 \times Int(U) & \longrightarrow & Y(0,T)^2 \\ (\bar{h}_1 = h_1 - h_{1,D}, \bar{h} = h - h_D, K) & \longrightarrow & \mathcal{R}(\bar{h}_1, \bar{h}, K) \end{array}$$

where

- $Z(0,T) = W(0,T) \cap L^\infty(0,T;L^2(\Omega)) \cap L^s(0,T;W^{1,s}(\Omega))$, $s > 4$
- $Y(0,T) = L^2(0,T;H^{-1}(\Omega)) \cap L^s(0,T;W^{-1,s}(\Omega))$
- $U = \{K \in BV(\Omega) \cap L^\infty(\Omega), K_m \leq K \leq K_M \text{ and } TV(K) \leq C\}$ with $c < C$, (c being the constant defining U_{adm}).

Differentiability of the operator $\mathcal{Q}$

Proposition

The mapping $\mathcal{Q}$ is continuous and differentiable from U_{adm} to $Z(0,T)$.

Proof:

- The differentiability of $\mathcal{R}(\bar{h}_1, \bar{h}, K)$ with respect to $(\bar{h}_1, \bar{h})$ on $Z(0,T)^2 \times Int(U)$ at the point $(\bar{h}_1, \bar{h}, K)$
- $D_{(\bar{h}_1(K), \bar{h}(K))}\mathcal{R}(\bar{h}_1, \bar{h}, K)$ is an isomorphism from $\left(Z(0,T)\right)^2$ to $\left(Y(0,T)\right)^2$, for all $K \in U_{adm}$.

Characterization of optimal control

Thanks to the well-posedness of the adjoint problem and the differentiability of the operator associating with K the state variables, we prove that the optimality system has at least one solution. We check that the required minimum, K^* satisfies optimality system .

Proposition

Let K^* be a solution of problem $(\mathcal{O})$, there exist a couple $(h_1^* - h_{1,D}, h^* - h_D) \in W(0,T)^2$ solution of $P(K^*)$ and a couple $\lambda^* = (\lambda_f^*, \lambda_i^*) \in W(0,T)^2$ solution of $R(K^*)$ such that, for all $K \in U_{adm}$,

$$D_K\mathcal{J}(K^*) \cdot (K(x) - K^*(x)) \geq 0. \tag{9}$$

with

$$\begin{aligned} D_K\mathcal{J}(K^*) \cdot K = \int_{\Omega_T} & K[(h_2 - h^*)(\alpha\nabla h^* + \nabla h_1^*) \cdot \nabla\lambda_i^* \\ & +\left((h_2^* - h_1^*)\nabla h_1^* + \alpha(h_2 - h^*)\nabla h^*\right) \cdot \nabla\lambda_f^*]\, dxdt. \end{aligned} \tag{10}$$

Characterization of optimal control

Proof :

- the application $K \longrightarrow \mathcal{J}(K) = \mathcal{L}(K, h_1(K), h(K), \lambda_f, \lambda_i)$ is differentiable with respect to K,

$$D_K\mathcal{J}(K^*)(\delta_K) = \int_{\Omega_T} \delta_K[(h_2 - h^*)(\alpha\nabla h^* + \nabla h_1^*)\cdot\nabla\lambda_i + ((h_2 - h_1^*)\nabla h_1^* + \alpha(h_2 - h^*)\nabla h^*)\cdot\nabla$$

$$+ \int_{\Omega_T} [-\phi\frac{\partial\lambda_i}{\partial t}\varphi_i + (\delta + \alpha K^*(h_2 - h^*))\nabla\lambda_i\cdot\nabla\varphi_i + \alpha K^*(h_2 - h^*)\nabla\lambda_f\cdot\nabla\varphi_i]$$

$$- \int_{\Omega_T} [K^*\nabla h_1^*\cdot\nabla\lambda_i + \alpha K^*\nabla h^*\cdot\nabla(\lambda_i + \lambda_f)]\varphi_i - \int_{\Omega_T}(h_{obs} - h^*)\varphi_i$$

$$+ \int_{\Omega_T} [-\phi\frac{\partial\lambda_f}{\partial t}\varphi_f + ((\delta + K^*(h_2 - h_1^*))\nabla\lambda_f)\cdot\nabla\varphi_f + K(x)(h_2 - h^*)\nabla\lambda_i\cdot\nabla\varphi$$

$$- \int_{\Omega_T} K^*\nabla h_1^*\cdot\nabla\lambda_f\,\varphi_f - \int_{\Omega_T}(h_{1,obs} - h_1^*)\varphi_f \quad \forall\delta_K \in U_{adm}$$

with $h^* = h(K^*)$, $h_1^* = h_1(K^*)$. Then taking $\lambda_f = \lambda_f^*$ and $\lambda_i = \lambda_i^*$ where $\lambda_i^* = \lambda_i(K^*, h_1^*, h^*)$, $\lambda_f^* = \lambda_f(K^*, h_1^*, h^*)$ is the unique solution of the adjoint problem leads to (10).

Description of the optimization algorithm

Aya Mourad, Thèse de doctorat, 2017.

- We first note that practically, we only have specific observations (in space and in time) corresponding to the number of monitoring wells, we thus adapt the previous results to the case of discrete observations data.
- Besides, K is discretized using the zonation method described in the paper:
  - N-Z Sun, Inverse problems in groundwater modelling, Kluwer Academic Publishers, Dordrecht, Netherland (1994).

Description of the optimization algorithm

Aya Mourad, Thèse de doctorat, 2017.

- We use of the limited memory variable metric algorithm on bound constrained optimization problem (BLMVM) developed by Benson and Moré.
  This algorithm uses projected gradients to construct a limited memory BFGS matrix and to determine a step direction.
  The step length is computed using the line search method which enforces it to satisfy the decrease condition and which also attempts to verify the curvature condition.
  - S.J. Benson, J. Moré, *A limited Memory variable-metric Algorithm for Bound-constrained Minimization*, Technical Report ANL/MSC-P909-O901, Mathematics and Computer Science Division, Argonne National Laboratory (2001).
  - Byrd, Richard H.; Lu, Peihuang; Nocedal, Jorge; Zhu, Ciyou, *A Limited Memory Algorithm for Bound Constrained Optimization*, SIAM Journal on Scientific and Statistical Computing. 16 (5): 1190â1208. doi:10.1137/0916069, (1995).

Description of the optimization algorithm

- The aquifer is figured by a parallelepiped $(x, y) \in [-50, 50] \times [-20, 20], z \in [-20, 0]$.
- In first step, we take K_{exact} for the exact value of the hydraulic conductivity, then the saltwater/freshwater interface depth h and the depth of the interface between dry zone and saturated zone h_1 are computed by solving the exact problem associated with this value of K_{exact}; these numerical values of h and h_1 have been considered as observed data.
- Then starting from an arbitrary initial estimate of this parameter, we compute the optimal solution by the parameters identification procedure.

Experiment 1

In the first experiment, the hydraulic conductivity is considered constant over the full domain of aquifer. The algorithm is applied for one well (case i), two wells (case ii) and three wells (case iii).

Table : Experiment 1.

case	number of wells	exact values	initial values	identified values
i	1	K= 50 m/d	K_0= 90 m/d	K_1= 51.371 m/d
ii	2	K= 50 m/d	K_0= 90 m/d	K_2= 50.064 m/d
iii	3	K= 50 m/d	K_0= 90 m/d	K_3= 50.001 m/d

Experiment 2

In the second experiment, we always consider that the hydraulic conductivity is constant over all of the domain but we only have observations concerning the elevation of the upper surface of the aquifer in three monitoring wells (case i). In (case ii), we have observations for the two interfaces depths.

Table : Experiment 2.

case	number of wells	exact values	initial values	identified values
i	3	K= 40 m/d	K_0= 85 m/d	K_1= 41.56 m/d
ii	1	K= 40 m/d	K_0= 85 m/d	K_2= 39.04 m/d

The comparison between these two results illustrates the importance to get information about each interface depth in order to obtain more accurate results.

Experiment 3

In the last experiment, we consider that the aquifer is split into three zones. We consider that we have one monitoring well in each zone in the first case (case i) and two monitoring wells in each zone in the second case (case ii).

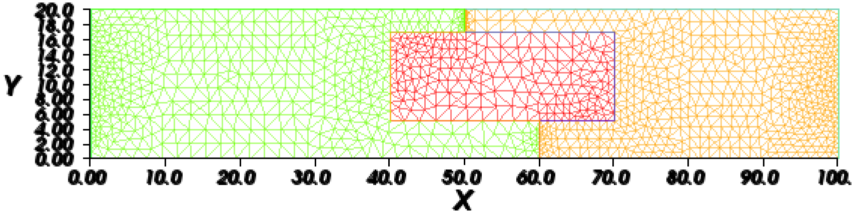


Figure : Schematisation of the aquifer in experiment 4.

Experiment 3

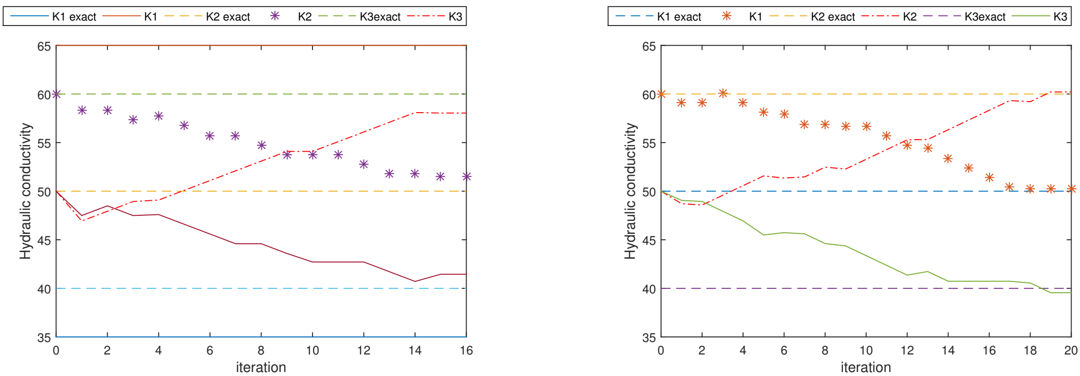


Figure : Graph representing the convergence of hydraulic conductivity in case i (on the left) case ii (on the right)

Experiment 3

Table : experiment 3.

case	number of wells	exact values	initial values	identified values
i	3	$K_1 = 50$ m/d	$K_1 = 60$ m/d	$K_1 = 50.45$ m/d
		$K_2 = 60$ m/d	$K_2 = 50$ m/d	$K_2 = 59.67$ m/d
		$K_3 = 40$ m/d	$K_3 = 50$ m/d	$K_3 = 40.14$ m/d
ii	6	$K_1 = 50$ m/d	$K_1 = 60$ m/d	$K_1 = 49.90$ m/d
		$K_2 = 60$ m/d	$K_2 = 50$ m/d	$K_2 = 60.10$ m/d
		$K_3 = 40$ m/d	$K_3 = 50$ m/d	$K_3 = 39.92$ m/d

Thank you for your attention !